from last time: $y_c = Y - y_p \Rightarrow Y = y_c + y_p$

can be shown that y_c is a solution to corresponding homogeneous eqn

since Y can be written as $Y = \underbrace{y_c}_{c_1y_1 + c_2y_2 + \dots + c_ny_n} + y_p \leftarrow$ LC

called a complementary function

tie all together:

thm: let y_p be a particular solution of non-homogeneous eq'n given linearly indep. solns $y_1, y_2, \dots, y_n$ of corresponding homogeneous eq'n

if Y is a sol'n of non-homogeneous eq'n CBW as $Y = c_1y_1 + c_2y_2 + \dots + c_ny_n + \boxed{y_p}$ *

ex. given $y'' + 4y = 12x$ s.t. $y_p = 3x$ and $y_c = c_1\cos 2x + c_2 \sin 2x$

non-HG

particular sol'n

complementary sol'n

find a solution that satisfies $y(0) = 5$, $y'(0) = 7$

gen sol'n CBW as $y(x) = y_c + y_p$

$y(x) = c_1 \cos 2x + c_2 \sin 2x + 3x$

$y(0) = c_1 + 0 + 0 = 5 \quad c_1 = 5$

$y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x + 3$

$y'(0) = 0 + 2c_2 + 3 = 7$

$c_2 = 2$

thus $y(x) = 5\cos 2x + 2\sin 2x + 3x$ is desired sol'n for non-HG eq'n based on initial conditions

Section 3.3 nth Order HG Eq'n w Constant Coefficients

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format: $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$

coefficients are $\mathbb{R}$ constants
$a_n \neq 0$

look for a <u>single</u> sol'n

explore $y = e^{rx}$

$\frac{dy}{dx} = re^{rx}$

$\vdots$

$\frac{d^k y}{dx^k} = r^k e^{rx}$

can see that each deriv. has a constant coefficient

goal: find r s.t. all the multiples of e^{rx} add to zero

recall characteristic eq'n:

when $y = e^{rx}$ $\quad ay'' + by' + cy = 0$ becomes

$ar^2 + br + c = 0$ $\quad 2^{nd}$ order eqn

expand def'n to apply to nth order HG eq'ns

char eqn $a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0$

<u>Solving Characteristic Eqn:</u>

as per Fundamental Theorem of Algebra, every nth degree polynomial has n roots not necessarily
- distinct
- real

beyond $n=2$ solving polynomial may be a nightmare

thm: if roots of a polynomial are real <u>and</u> distinct then $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots + c_n e^{r_n x}$ is a gen. sol'n.

ex. solve $y''' + 3y'' - 10y' = 0$ given $y(0) = 7$
$y'(0) = 0$
$y''(0) = 70$

$y'(0)=0$

$y''(0)=70$

char eqn : $r^3+3r^2-10r=0$

$r(r^2+3r-10)=0$

$r(r+5)(r-2)=0$

$r_1=0 \quad r_2=-5 \quad r_3=2$

gen sol'n : $y(x)=c_1e^0+c_2e^{-5x}+c_3e^{2x}$

$$\left.\begin{aligned}
y(0) &= c_1+c_2+c_3=7\\
y'(x) &= -5c_2e^{-5x}+2c_3e^{2x}\\
y'(0) &= -5c_2+2c_3=0\\
y''(x) &= 25c_2e^{-5x}+4c_3e^{2x}\\
y''(0) &= 25c_2+4c_3=70
\end{aligned}\right]$$

$c_1=0$

$c_2=2$

$c_3=5$

$\therefore$ particular sol'n is $\boxed{y(x)=2e^{-5x}+5e^{2x}}$

Repeated Real Roots

thm: if char. eq'n has a repeated root r with multiplicity of k then relevant piece of gen sol'n would be:

$$e^{rx}(\underbrace{c_1+c_2x}_{\text{seen previously}}+c_3x^2+\ldots+c_kx^{k-1})$$

then the set of k functions $e^{rx}, xe^{rx}, x^2e^{rx}, \ldots, x^{k-1}e^{rx}$ are linearly independent on $\mathbb{R}$

ex. find gen sol'n of $9y^{(5)}-6y^{(4)}+y^{(3)}=0$

char eqn $9r^5-6r^4+r^3=0$

$r^3(9r^2-6r+1)=0$

$r^3(9r^2-3r-3r+1)=0$

$r^3(3r(3r-1)-1(3r-1))=0$

$$r^3\left(3r(3r-1)-1(3r-1)\right)=0$$
$$r^3(3r-1)(3r-1)=0$$
$$\underset{\substack{\downarrow\\ r_1=0\\ \text{mult is } 3}}{r^3}\ \underset{\substack{\downarrow\\ r_2=\frac{1}{3}\\ \text{mult is } 2}}{(3r-1)^2}=0$$

$$y(x)=\overbrace{c_1e^{0x}+c_2xe^{0x}+c_3x^2e^{0x}}+\overbrace{c_4e^{1/3x}+c_5xe^{1/3x}}$$
$$\boxed{y(x)=c_1+c_2x+c_3x^2+c_4e^{x/3}+c_5xe^{x/3}}$$

Complex Roots

recall MacLaurin series $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$

let $x=i\theta \longrightarrow e^{i\theta}$ $=1+i\theta+\frac{i^2\theta^2}{2!}+\frac{i^3\theta^3}{3!}+\frac{i^4\theta^4}{4!}+\frac{i^5\theta^5}{5!}+\dots$

$i=\sqrt{-1}$
$i^2=-1$
$i^3=-i$
$i^4=1$

$$=1+i\theta-\frac{\theta^2}{2!}-\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{i\theta^5}{5!}+\dots$$

regroup by real/imag.

$$=\underbrace{\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots\right)}_{\text{real component}}+i\underbrace{\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\dots\right)}_{\text{imaginary component}}$$

recall $\sin\theta=\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}$
$=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\dots$

$\cos\theta=\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}$
$=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\dots$

Euler's Formula: $e^{i\theta}=\cos\theta+i\sin\theta$

define e^z where z is an arbitrary complex number or $z=\overset{a+bi}{x+iy}$

can conclude that appearance of complex roots of char eqn will result in complex-valued sol'ns of DE

write as: $F(x)=f(x)+ig(x)$ where f,g themselves are real-valued functions

constructing $a+bi$ format

constructing $a+bi$ format

assuming no differentiability issues,
differentiate each separate term: $F'(x) = f'(x) + ig'(x)$

revisit e^{rx}: $F(x) = e^{rx}$ where $r = a \pm bi$

then $e^{(a+bi)x} = e^{ax}e^{bix}$ AND $e^{(a-bi)x} = e^{ax}e^{-bix}$

$y_1(x) = e^{ax}(\cos bx + i\sin bx)$ $\quad$ $y_2(x) = e^{ax}(\cos bx - i\sin bx)$

write 2 solns as a LC:

$$\begin{aligned}
y(x) &= C_1 e^{r_1 x} + C_2 e^{r_2 x} \\
&= C_1 e^{(a+bi)x} + C_2 e^{(a-bi)x} \\
&= C_1 e^{ax}(\cos bx + i\sin bx) + C_2 e^{ax}(\cos bx - i\sin bx) \\
&= \underline{C_1 e^{ax}\cos bx} + \underline{\underline{iC_1 e^{ax}\sin bx}} + \underline{C_2 e^{ax}\cos bx} - \underline{\underline{iC_2 e^{ax}\sin bx}} \\
&= (C_1 + C_2)e^{ax}\cos bx + i(C_1 - C_2)e^{ax}\sin bx
\end{aligned}$$

let $c_1 = C_1 + C_2$ $\qquad$ let $c_2 = i\,C_1 - C_2$

then $y(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$ is gen sol'n
where char. eq'n. has unrepeated complex roots
in the form $a+bi$ s.t $b \neq 0$

ex. given $y'' + b^2 y = 0$ where $b > 0$
char eqn: $r^2 + b^2 = 0$

$\sqrt{r^2} = \sqrt{-b^2} \Rightarrow r = \sqrt{b^2}\sqrt{-1}$

$r = \pm bi \longrightarrow \overset{a}{0} \pm bi$

$\therefore$ gen soln is $y(x) = \overset{e^{0x}}{\downarrow}(c_1 \cos bx + c_2 \sin bx)$

$$\boxed{y(x) = c_1 \cos bx + c_2 \sin bx}$$

ex. find particular sol'n given $y'' - 4y' + 5y = 0$ when $y(0) = 1$ $y'(0) = 5$

$r^2 - 4r + 5 = 0$

ex. find particular sol'n given $y'' - 4y' + 5y = 0$ when $y(0) = 1$ $y'(0) = 5$

$$r^2 - 4r + 5 = 0$$
$$(r-2)^2 + 1 = 0$$
$$(r-2)^2 = -1$$
$$r - 2 = \pm i$$
$$r = 2 \pm i$$

gen sol'n: $y(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$

$y(0) = c_1 = 1$

$\vdots$ finish on your own $c_2 = 3$

$\therefore \boxed{y(x) = e^{2x}(\cos x + 3 \sin x)}$

Repeated Complex Roots

ex. find gen. sol'n of HG DE whose char. eq'n has roots of:

$$\underbrace{-5, -5,}_{\text{mult: } 2} \underbrace{0, 0, 0, 0,}_{\text{mult: } 4} \underbrace{3,}_{m: 1} \underbrace{2 \pm 3i, 2 \pm 3i}_{\text{mult: } 2}$$

$$y(x) = \underbrace{c_1 + c_2 x + c_3 x^2 + c_4 x^3}_{\substack{\text{root} = 0 \\ \text{mult: } 4}} + \underbrace{c_5 e^{3x}}_{\substack{\text{root} = 3 \\ \text{mult: } 1}} + \underbrace{c_6 e^{-5x} + c_7 x e^{-5x}}_{\substack{\text{root} = -5 \\ \text{mult: } 2}} + \underbrace{e^{2x}(c_8 \cos 3x + c_9 \sin 3x) + x e^{2x}(c_{10} \cos 3x + c_{11} \sin 3x)}_{\substack{\text{root} = 2 \pm 3i \\ \text{mult: } 2}}$$